Types of Functions

A **polynomial function** of degree n is a function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (a_0 \neq 0)$$

where $a_0, a_1, \ldots, a_n$ are real constants and n is a nonnegative integer.

EXAMPLES:

$f(x) = 4x^3 - 2x^2 + 10x - 5$ (cubic)

$g(x) = 2x^4 - 3x + 1$

$h(x) = x^2 - 3x + 1$ (quadratic)

Types of Functions

A **rational function** is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions.

EXAMPLES:

$$F(x) = \frac{3x^3 + x^2 - x + 1}{x - 2}$$

$$G(x) = \frac{x^2 + 1}{x^2 - 1}$$

Types of Functions

A **power function** is a function of the form

$$f(x) = x^r$$

where r is any real number.

EXAMPLES:

$$f(x) = x^3$$

$$g(x) = \sqrt{x} = x^{1/2}$$

$$h(x) = \frac{1}{x^5} = x^{-5}$$

Exponential Functions

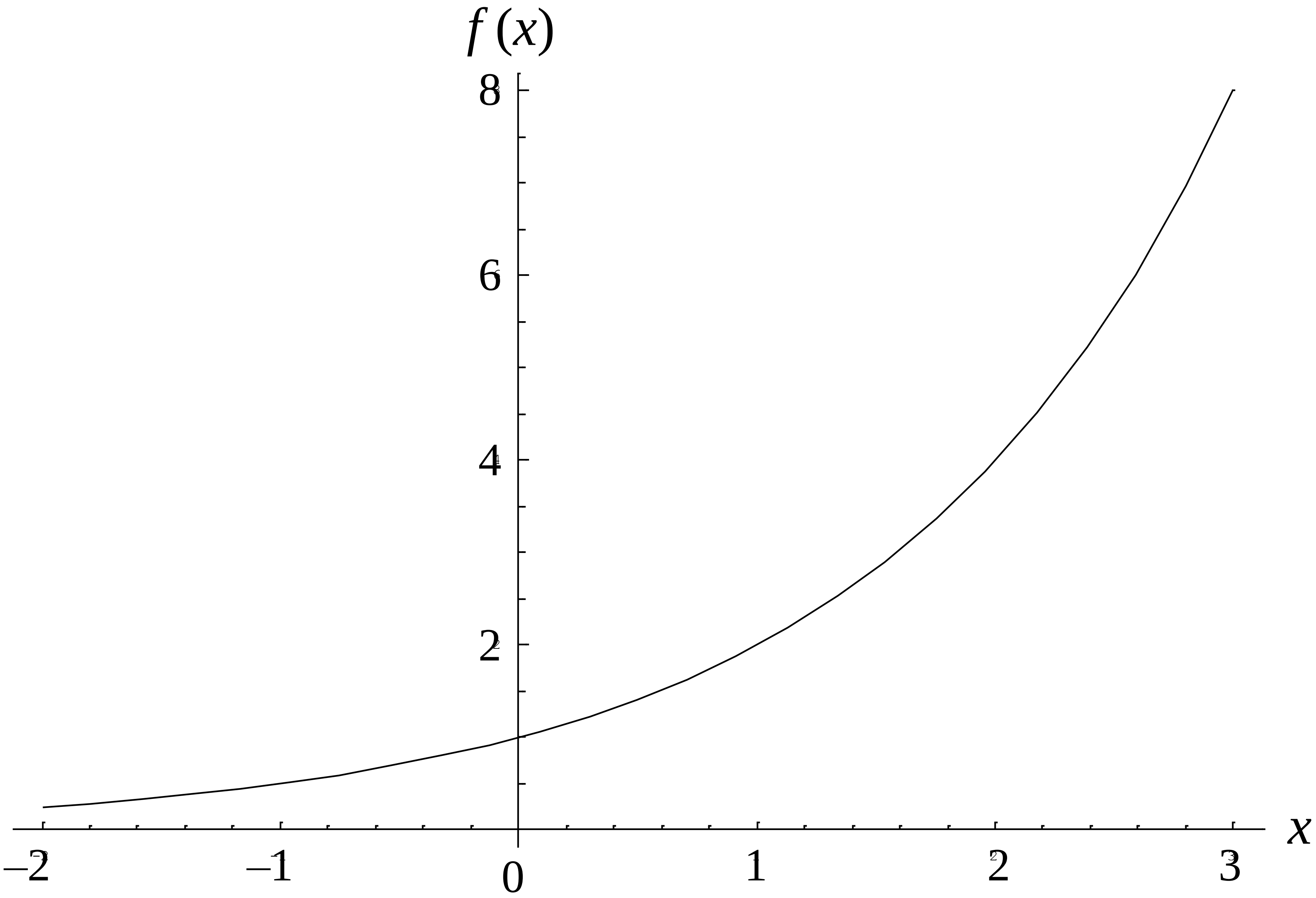
The function defined by

$$f(x) = b^x \qquad (b > 0, b \neq 1)$$

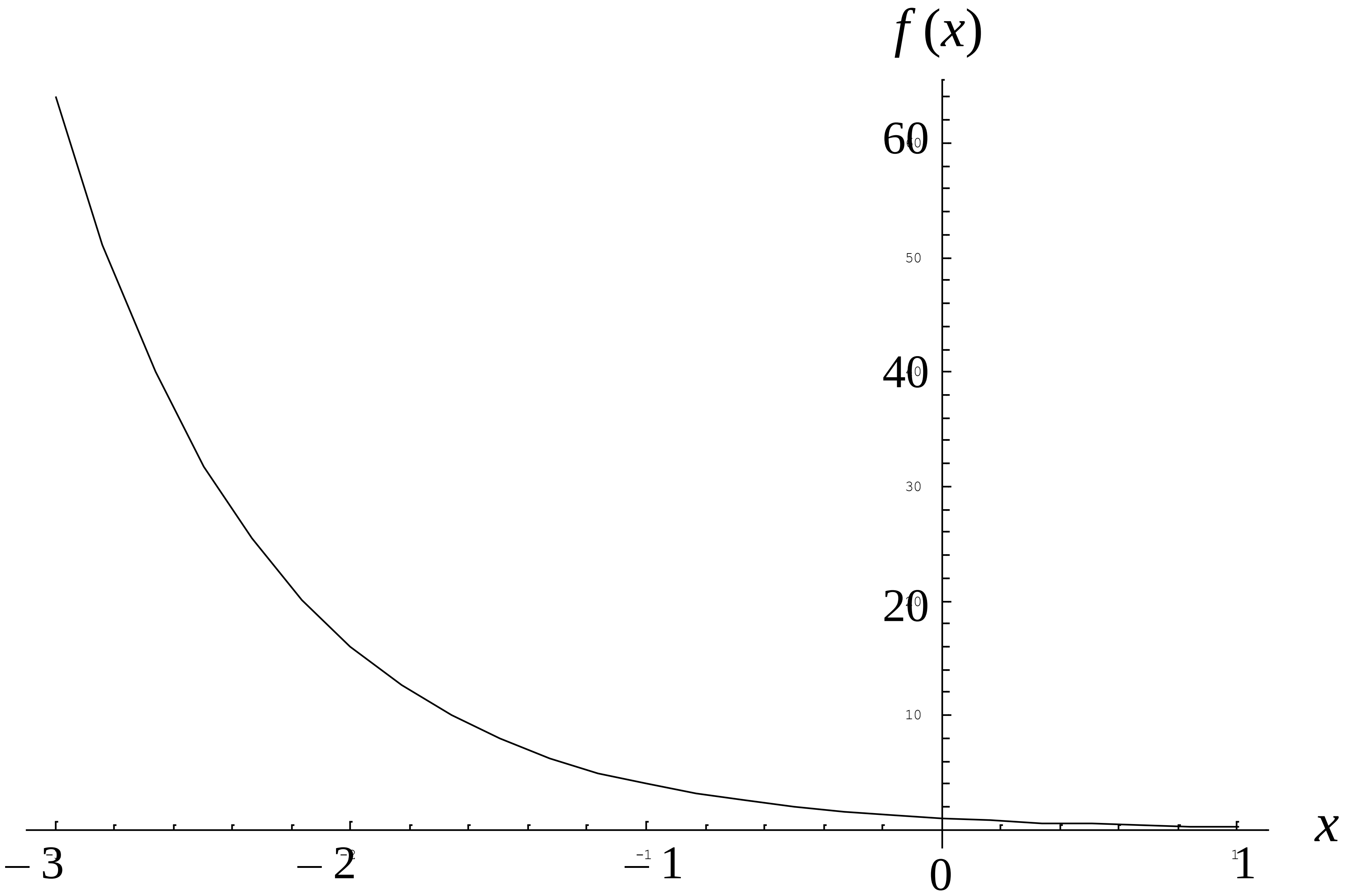
is called an **exponential function with base b and exponent x**. The domain of f is the set of all real numbers.

To see what this function looks like when graphed, consider the following two examples.

Example : $f(x) = 2^x$



Example : $f(x) = (1/4)^x$



Properties of the Exponential Function

The exponential function $y = b^x$ $(b > 0, b \neq 1)$ has the following properties:

1. Its domain is all real numbers, $(-\infty, \infty)$.

2. Its range is all positive real numbers, $(0, \infty)$.

3. Its graph passes through the point $(0, 1)$.

4. It is continuous on $(-\infty, \infty)$.

5. If $b > 1$, it is increasing on $(-\infty, \infty)$.

If $0 < b < 1$, it is decreasing on $(-\infty, \infty)$.

6. If $b > 1$, then $\lim_{x\to-\infty} b^x = 0$ and $\lim_{x\to\infty} b^x = \infty$.

If $0 < b < 1$, then $\lim_{x\to-\infty} b^x = \infty$ and $\lim_{x\to\infty} b^x = 0$.

Laws of Exponents

1. $b^x \cdot b^y = b^{x+y}$

2. $\dfrac{b^x}{b^y} = b^{x-y}$

3. $(b^x)^y = b^{xy}$

4. $(ab)^x = a^x b^x$

5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Examples

$2^{1/3} \cdot 2^{5/3} = ?$

$\dfrac{4^{13}}{4^{10}} = ?$

$(5^{1/3})^{-6} = ?$

$(5x)^3 = ?$

$\left(\dfrac{8}{27}\right)^{1/3} = ?$

Exercise

Simplify the expression $\dfrac{\left(3x^2y^{1/3}\right)^6}{x^3y^5}$.

Solve for x in the expression $4^{7x+1} = 2^{4x-2}$.

The Base e

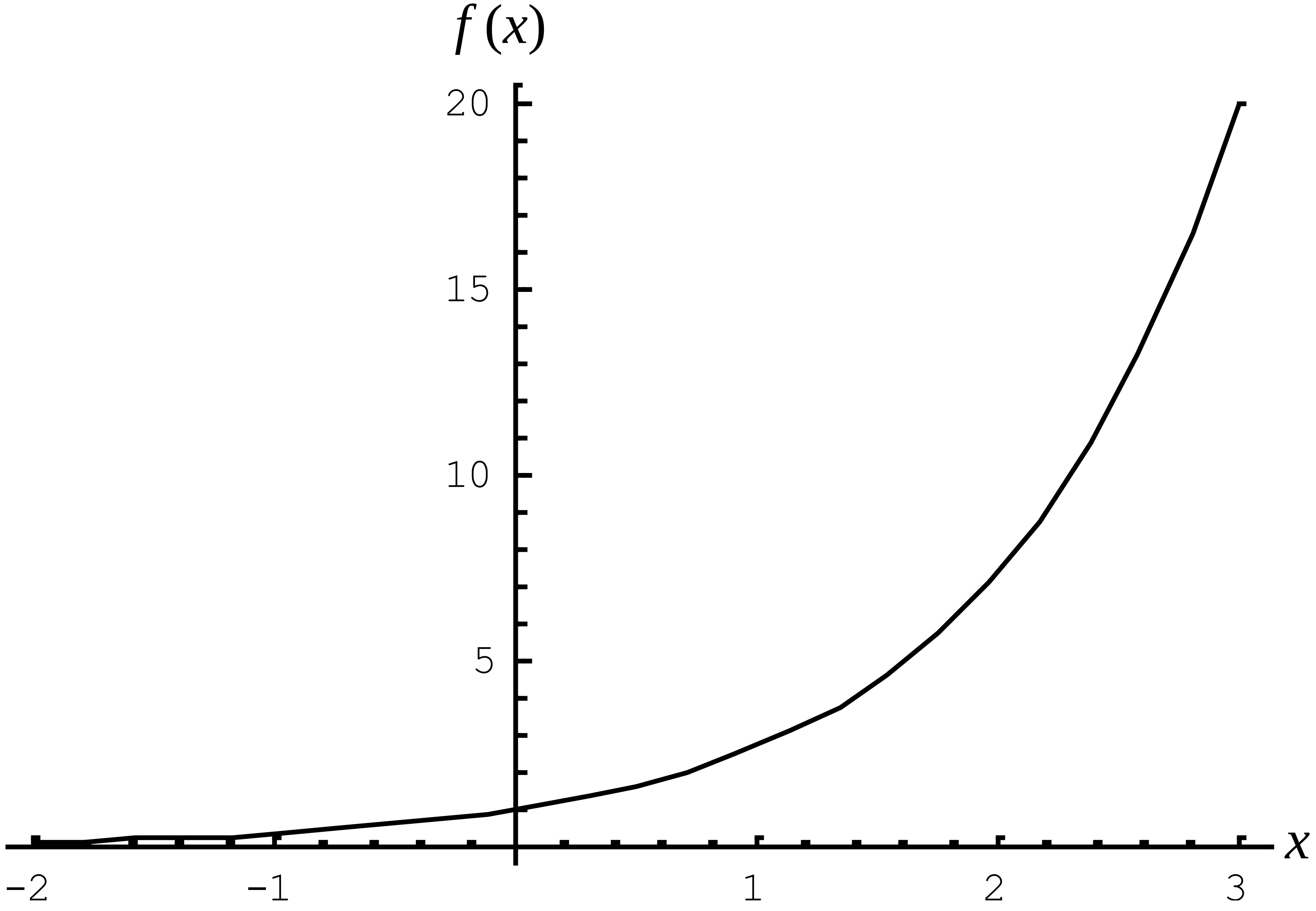
There is one base that arises naturally in many practical applications. This is the base e and it is defined by the following limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

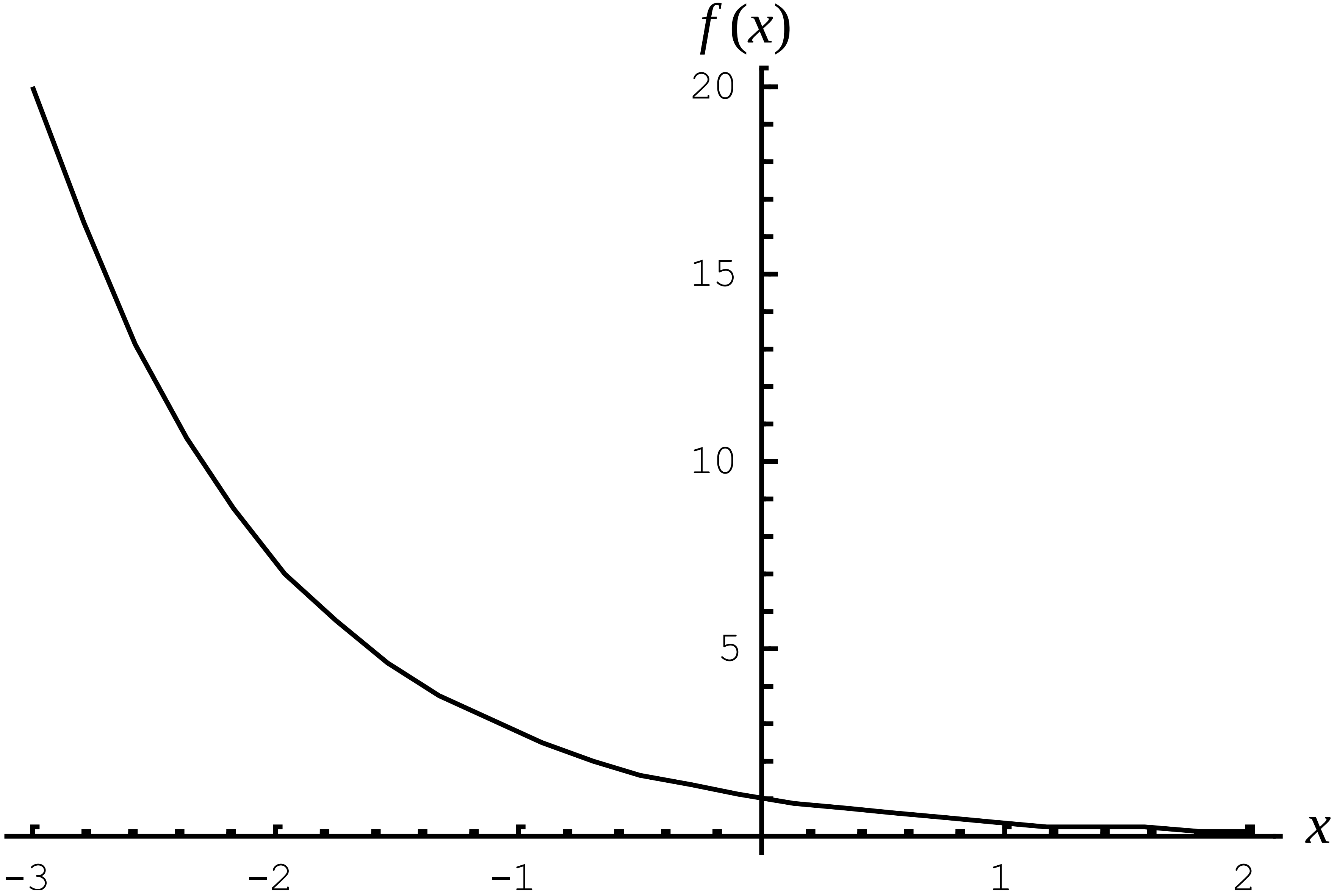
e is an irrational number whose value is approximately $e = 2.7182818\ldots$

The function $f(x) = e^x$ is called the **natural exponential function**.

Example: $f(x) = e^x$



Example: $f(x) = e^{-x}$



Logarithmic Functions

Consider the function

$$x = b^y \qquad (b > 0,\ b \neq 1).$$

Then the value y is called the **logarithm of x with base b** and is denoted by $y = \log_b x$.

In other words, $y = \log_b x$ if and only if $x = b^y$.

$\log_b x =$ the power you would need to raise b to in order to get an answer of x.

Note that x must necessarily be positive.

To see what this function looks like when graphed, consider the following examples.

Properties of the Logarithmic Function

The logarithmic function $y = \log_b x \ \ (b > 0, b \neq 1)$ has the following properties:

1. Its domain is all positive real numbers, $(0, \infty)$.

2. Its range is all real numbers, $(-\infty, \infty)$.

3. Its graph passes through the point $(1, 0)$.

4. It is continuous on $(0, \infty)$.

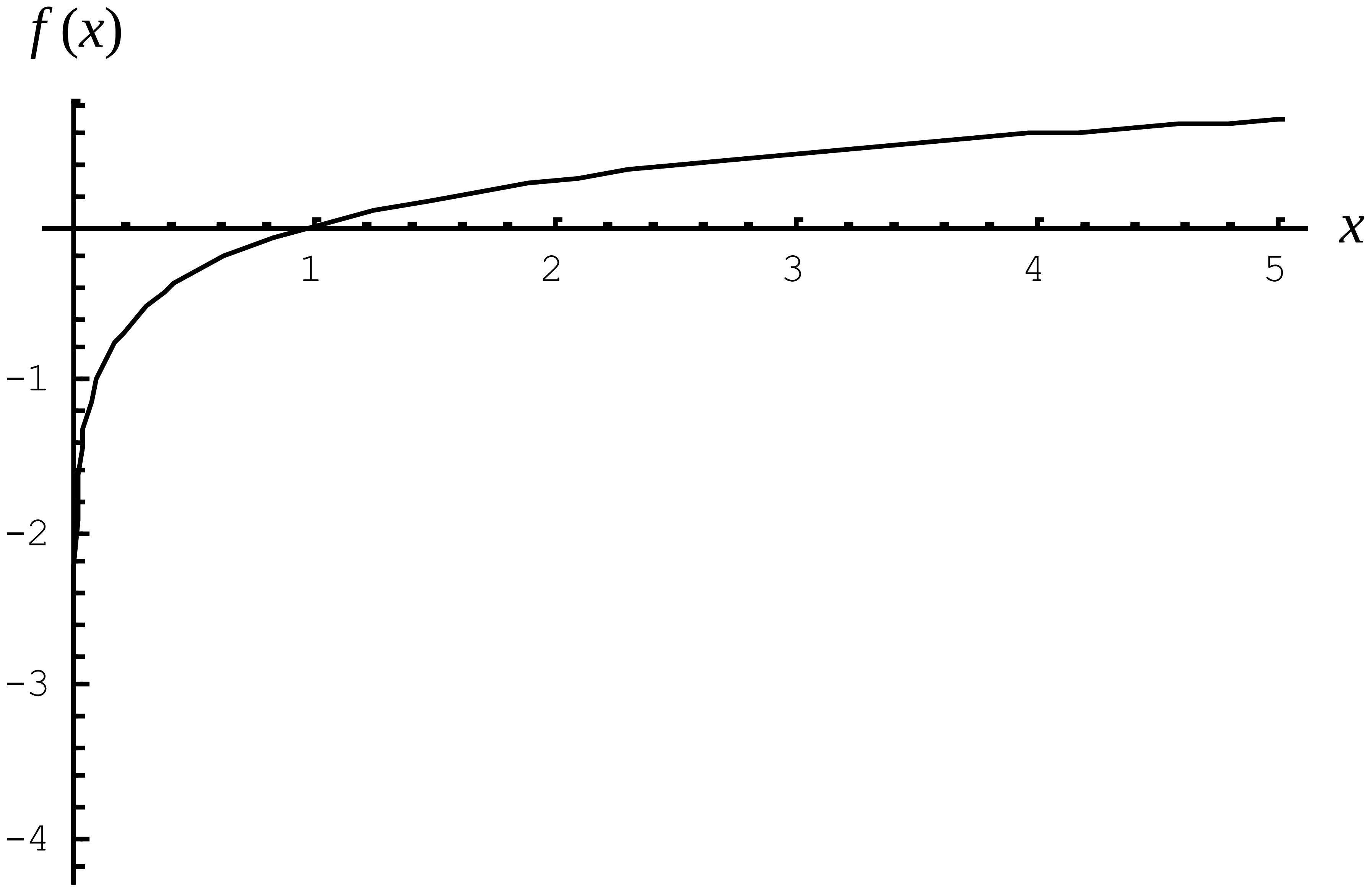
5. If $b > 1$, it is increasing on $(0, \infty)$.

If $0 < b < 1$, it is decreasing on $(0, \infty)$.

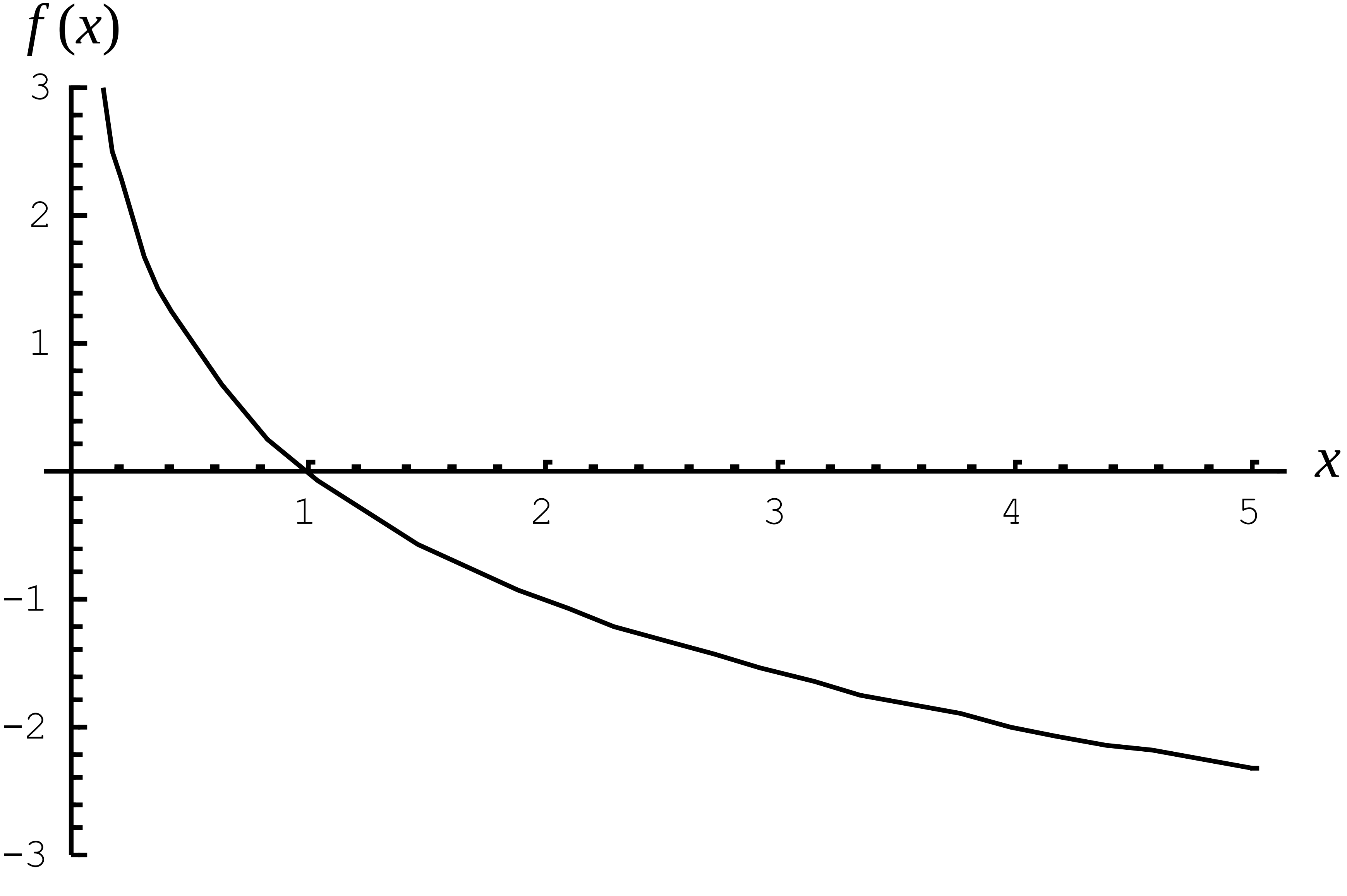
6. If $b > 1$, then $\lim\limits_{x \to 0^+} y = -\infty$ and $\lim\limits_{x \to \infty} y = \infty$.

If $0 < b < 1$, then $\lim\limits_{x \to 0^+} y = \infty$ and $\lim\limits_{x \to \infty} y = -\infty$.

<u>Example</u>: $f(x) = \log_{10} x$



Example: $f(x) = \log_{1/2} x$



Examples: Graphing Logarithmic Equations

Example 1:

Graph the following function: $g(x) = 3\log_2(x-1) - 2$

Solution:

Parent function $f(x) = \log_2(x)$

$g(x) = 3[f([x-1])] - 2$

$a = 3, b = 1, c = -1, d = -2$

We generate our table of values for the parent function:

x	$f(x)$
0	$Asymptote$
$1/2$	-1
1	0
2	1
4	2

We then apply the transformations:

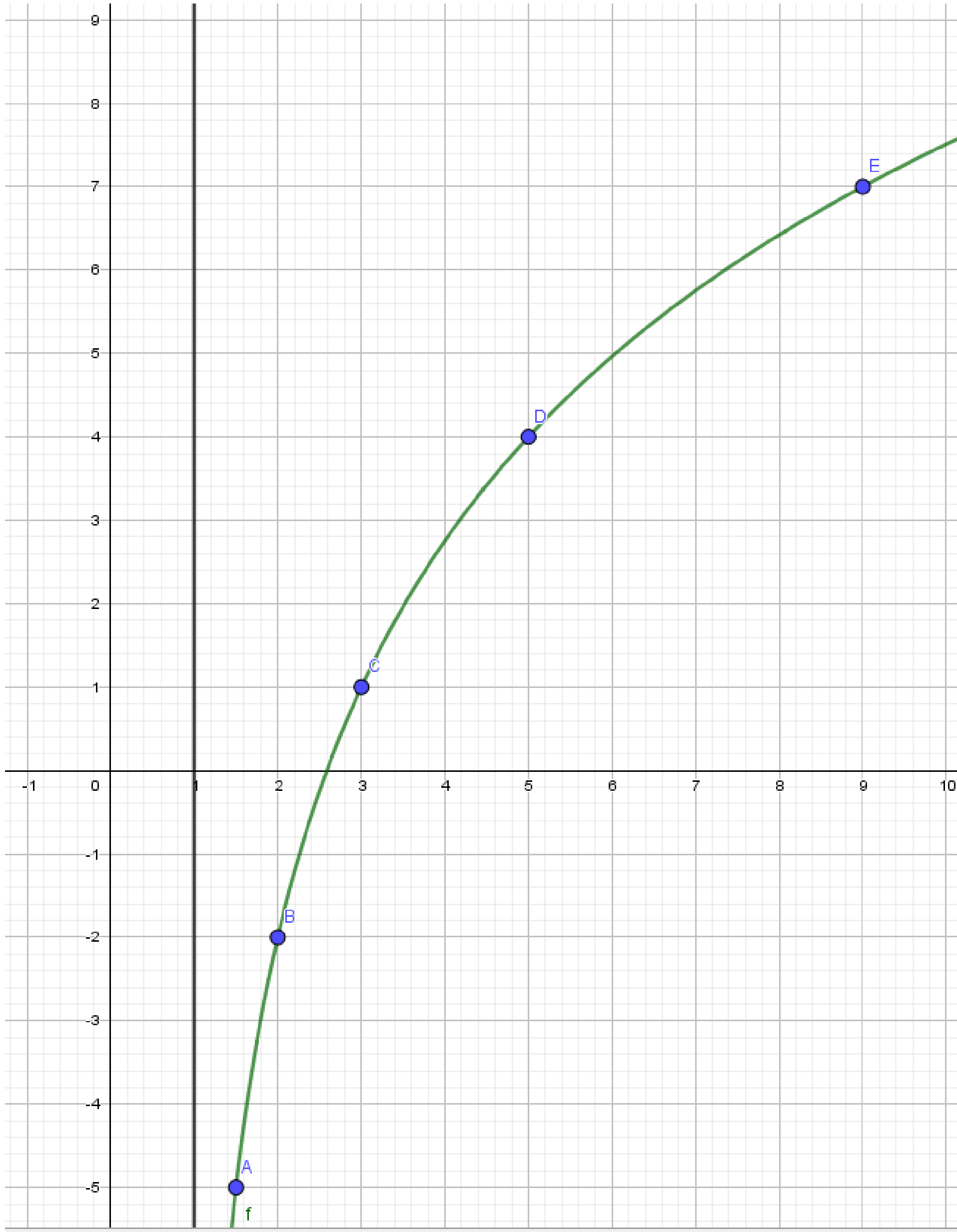
New x values:

$\frac{x}{b} - c \rightarrow \frac{x}{1} + 1$

New y values:

$(y \times a) + d \rightarrow 3y - 2$

New x	$g(x)$
$\frac{0}{1} + 1 = 1$	$Asymptote$
$\frac{1/2}{1} + 1 = 1.5$	$3(-1) - 2 = -5$
$\frac{1}{1} + 1 = 2$	$3(0) - 2 = -2$
$\frac{2}{1} + 1 = 3$	$3(1) - 2 = 1$
$\frac{4}{1} + 1 = 5$	$3(2) - 2 = 4$



Formulas: Log Laws

A log law is a formula used to simply expressions using logs. When working to simplify logs, it helps to get all of the arguments inside of the brackets to match the base of the log (if possible).

Formula(s):	**When To Use it:**	
$\log_a(a^x) = x$ $a^{\log_a(x)} = x$ $\log_a(bc) = \log_a(b) + \log_a(c)$ $\log_a(b/c) = \log_a(b) - \log_a(c)$ $\log_a(b^c) = c(\log_a(b))$ $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$	**1) When simplifying expressions involving log.** **2) When solving exponential or log equations.** **3) When working with exponential equations and you want to change the format from powers and products to multiplication and addition. (We will see this later)**	

The Natural Logarithm

The function $g(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.

Therefore, we have the following properties:

1) $\ln e = 1$

2) $\ln e^x = x$ (for any real number x)

3) $e^{\ln x} = x$ $(x > 0)$

Exercise

Determine the value of the following terms:

a) $\log_3 27$

b) $\log_7 1$

c) $\log_{1/3} 9$

d) $\log_{10} 10$

Exercise

Solve for x in the following expressions:

a) $\log_2 x = 5$

b) $\log_{27} 3 = x$

c) $\log_x 9 = 2$

Exercise

Use the Laws of Logarithms to simplify the following expressions:

a) $\log x(x+1)^4$

b) $\log \dfrac{\sqrt{x+1}}{x^2+1}$

c) $\log_5 \dfrac{25x^7 y}{\sqrt{z}}$

Examples: Using Log Laws

Example :

Simplify the following:

a) $3\log_2(\sqrt{8})$

b) $\ln\left(e^{e^x}\right)$

c) $\dfrac{\left[\ln(35)+\ln\left(\frac{1}{7}\right)\right]}{\ln(25)}$

Solution:

a) $3\log_2(\sqrt{8}) = 3\log_2\left(2^{3/2}\right) = 3\left(\frac{3}{2}\right) = \frac{9}{2}$

b) $\ln\left(e^{e^x}\right) = e^x$

c) $\dfrac{\left[\ln(35)+\ln\left(\frac{1}{7}\right)\right]}{\ln(25)} = \dfrac{\left[\ln(5(7))+\ln\left(7^{-1}\right)\right]}{\ln(5^2)} = \dfrac{[\ln(5)+\ln(7)-\ln(7)]}{2\ln(5)} = \dfrac{[\ln(5)]}{2\ln(5)} = \dfrac{1}{2}$

Examples: Using Log Laws

Example 3:

Rewrite the expression in terms of ln: $\frac{\log_2(x)}{\log_4(x)}$

Solution:

Using our change of base formula we get:

$$
\begin{aligned}
\frac{\log_2(x)}{\log_4(x)} &= \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln x}{\ln 4}\right)} \\
&= \left(\frac{\ln x}{\ln 2}\right) \times \left(\frac{\ln 4}{\ln x}\right) \\
&= \frac{\ln 4}{\ln 2} \\
&= \frac{\ln 2^2}{\ln 2} \\
&= \frac{2\ln 2}{\ln 2} \\
&= 2
\end{aligned}
$$

We note that the original expression had x and our final expression does not. This means that it evaluates to a constant for all $x \in (0, \infty)$

Examples: Using Log Laws

Example 4:

Solve the following equation: $4e^{3k} - 6 = 10$

Solution:

We work to isolate k one step at a time:

$4e^{3k} - 6 = 10$

$4e^{3k} = 16$

$e^{3k} = 4$

$\ln\left(e^{3k}\right) = \ln(4)$

$3k = \ln(4)$

$k = \frac{\ln(4)}{3}$

Note that **this does not simplify to become $\mathbf{ln}\left(\frac{\mathbf{4}}{\mathbf{3}}\right)$** as we have no such log law!!

Examples: Using Log Laws

Example 5:

Solve the following equation: $\log_2(-10+x)+\log_2(-x)=4$

Solution:

Using exponent laws, we simplify the expression, then we work to solve by getting rid of log:

$\log_2(-10+x)+\log_2(-x)=4$
$\log_2(10x-x^2)=4$
$2^{\log_2(10x-x^2)}=2^4$
$10x-x^2=16$
$x^2-10x+16=0$
$(x-8)(x-2)=0$

$\therefore x=8$ or $x=2$, however, since neither of these solutions can be put into the domain of the second log, it turns out that <u>this log equation has no solution</u>.

Exercise

Use the Laws of Logarithms to simplify the following expression:

$$\ln \frac{e^x}{1+e^x}$$

Exercise

Solve for t in the following equation:

$$\frac{200}{1+3e^{-0.3t}} = 100$$

Definition: Inverse Function

An <u>inverse function of $f(x)$</u> is a function (denoted by $\underline{f^{-1}(x)}$) that has the property $\underline{f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x}$ for all $\underline{x}$ in the domain of f (or f^{-1} depending)

Important Note: We need f^{-1} to be a function. Thus, we may need to give a restriction on the domain of f to allow for f^{-1} to be a function.

<u>Example 1:</u>

Are $f(x) = x$ and $g(x) = \frac{1}{x}$ inverses of each other?

<u>Solution:</u>

We can check:

$f\left(g(x)\right) = \frac{1}{x} \neq x$ thus these functions are not inverses.

Examples: Inverse Function

Example 2:

Are $f(x) = 3x$ and $g(x) = \frac{1}{3}x$ inverses of each other?

Solution:

We can check: $f(g(x)) = 3\left(\frac{1}{3}x\right) = x$

$g(f(x)) = \frac{1}{3}(3x) = x$

Thus these functions are inverses of each other.

Example 3:

Are $f(x) = x^2$ and $g(x) = \sqrt{x}$ inverses of each other?

Solution:

We may think these are inverses, but if we check: $g(f(x)) = \sqrt{x^2}$

We note that if $x = -1$ we get:

$$g(f(x)) = \sqrt{(-1)^2}$$

$$= \sqrt{1} \quad \neq -1 = x$$

Thus we do not have all values of x work for this inverse (positive numbers would work, but negative numbers would not). If we restrict the domain of $g(x)$ to be $[0, \infty)$ then we would have these two functions are inverses.

Strategy: Finding Inverses Graphically by switching coordinates

How To Use it:	When To Use it:	Why this works?
1) Select key points (x, y) on the original graph (vertex, intercepts, endpoints, etc..) 2) Switch the coordinates of the points to become (y, x) 3) Plot the new points to construct the graphical representation of the inverse i.e. graphically reflect the graph about the line y=x. 4) If the questions requests for an inverse function, then restrict the domain of the original that will allow the inverse graph to be a function (pass the vertical line test).	When finding the inverse of a function when given a graph (alternatively if you are given an equation, you can graph the function and find the inverse graph as well).	When finding an inverse function of $f(x)$, we want to find a function $g(x)$ such that $g\big(f(x)\big) = x$ This means that if we have a point on f called $\big(x, f(x)\big)$ then this point will appear in g as $\big(x, f(x)\big)$ which simply switches the coordinates.

Example 4:

How can we look at a graph and know that it's inverse will be a function?

Solution:

If the function passes the horizontal line test, then its inverse will be a function (as a horizontal line in the original function will represent a vertical line in the inverse when we switch x and y).

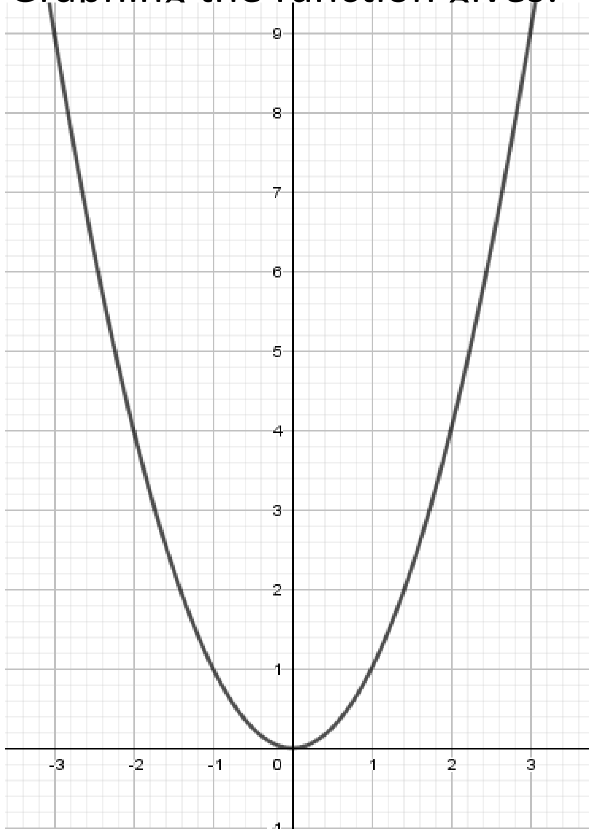
Examples: Finding Inverses Graphically

Example 5:

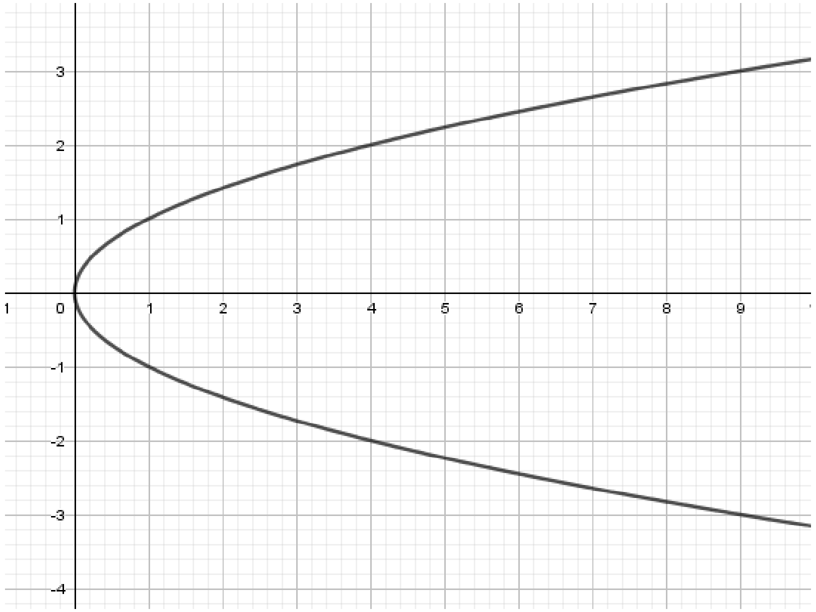
Determine the graphical representation of the inverse of $f(x) = x^2$. Is it a function? If it is not a function, explain how you can restrict the domain of $f(x)$ to make the inverse a function.

Solution:

Graphing the function gives:



When choosing key points, and switching the coordinates, it gives us the graph:

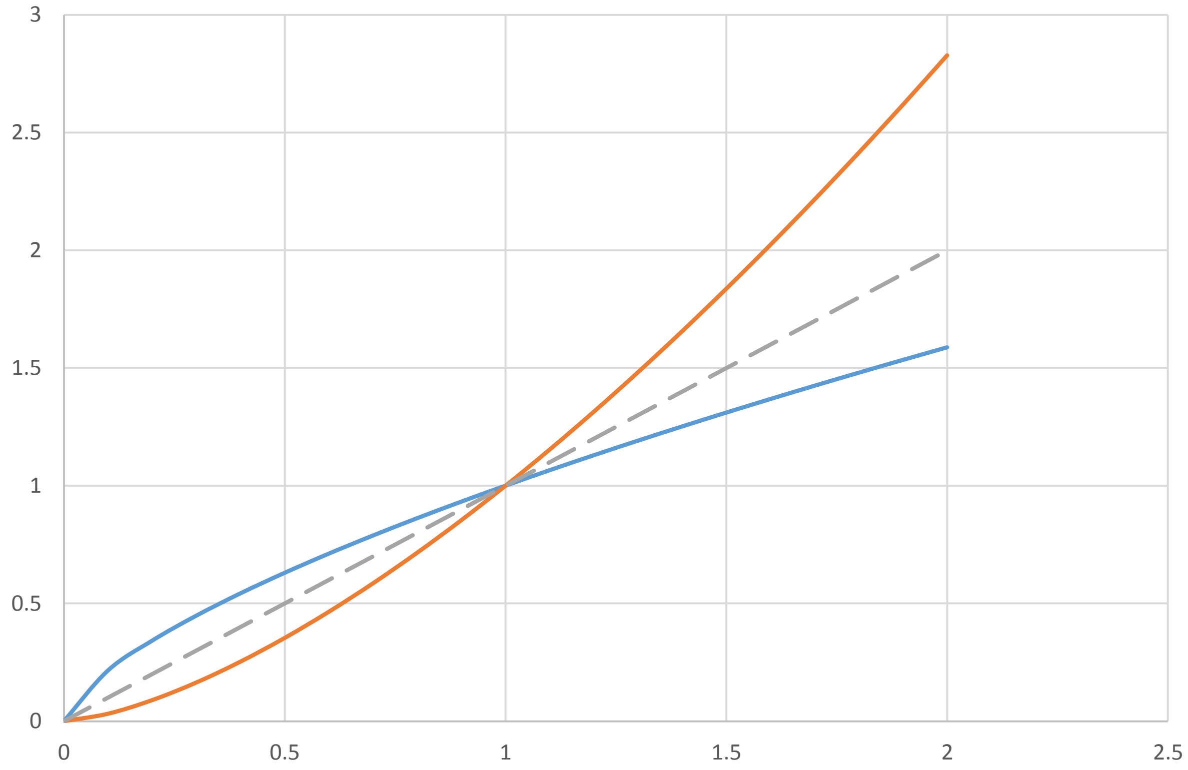


Since it does not pass the vertical line test, this inverse is not a function.

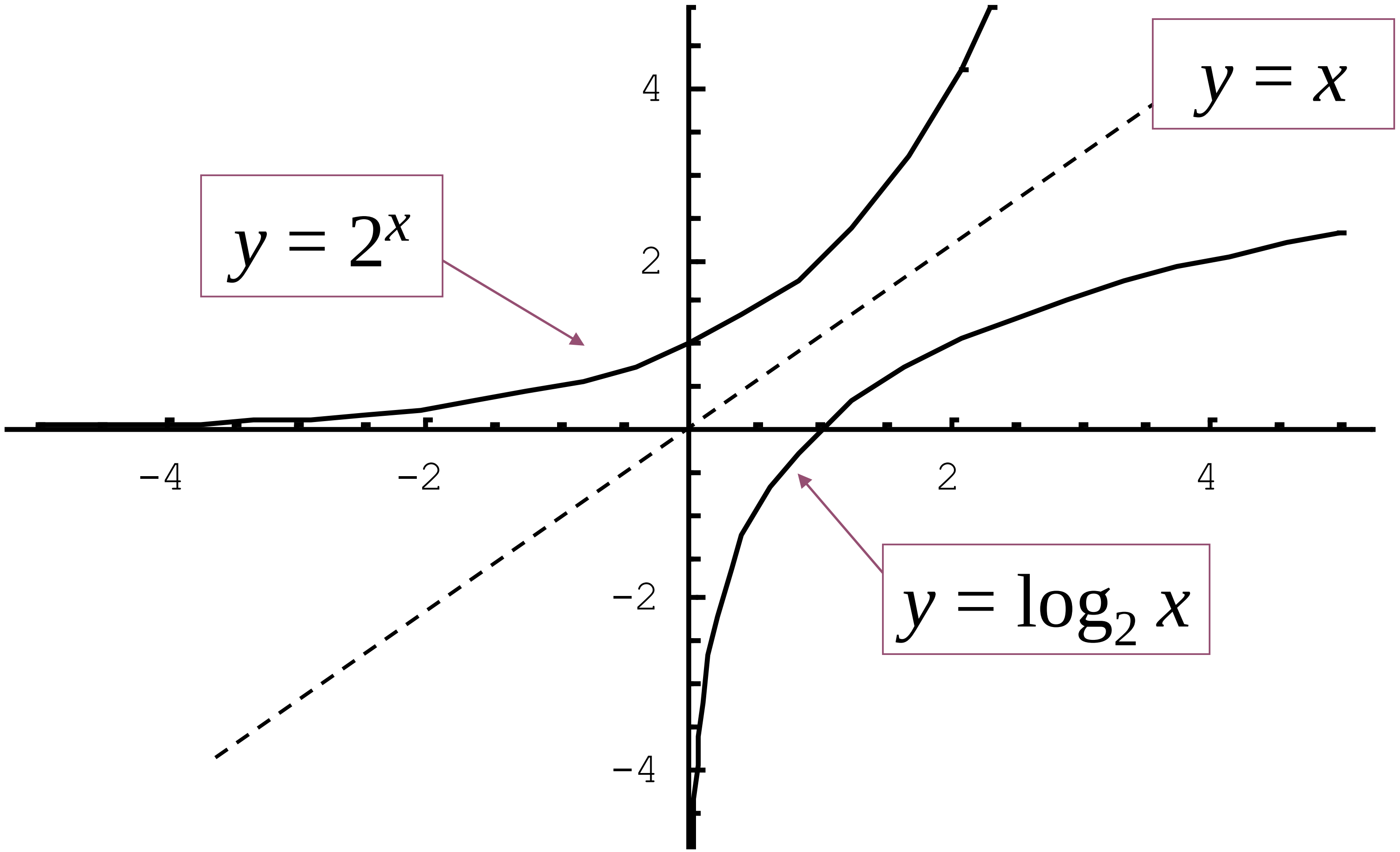
If we restrict the domain of the original to be $[0, \infty)$ or $(-\infty, 0]$, we will have that the inverse will be a function.

Examples: Finding Inverses Graphically

Draw the graph of $f(x) = x^{2/3}\ for\ x \geq 0$ and find $f^{-1}(x)$ graphically.



Exponential and Logarithmic Function Graphs



Strategy: Finding Inverses Algebraically by switching coordinates

How To Use it:	When To Use it:	Why this works?
1) Rewrite $f(x)$ as y. 2) Switch x for y (and vice versa) in the equation. 3) Solve for y. 4) Replace y with $f^{-1}(x)$ 5) If the relation is not a function, impose a domain restriction on the original function (not the inverse function) so that the inverse will be a function. Relations that are not functions usually are things that include $\pm\sqrt{\ }$.	When you want to find the inverse function given an equation.	When finding an inverse function of $f(x)$, we want to find a function $g(x)$ such that $g\big(f(x)\big) = x$ This means that if we have a point on f called $\big(x, f(x)\big)$ then this point will appear in g as $\big(x, f(x)\big)$ which simply switches the coordinates. The remainder is simply solving for the output (as functions are written in the form $f(x) = \cdots$

Determine the inverse function of the following: $f(x) = \frac{3x+2}{x-1}$

Solution:

1) $y = \frac{3x+2}{x-1}$

2) $x = \frac{3y+2}{y-1}$

3) $x(y-1) = 3y+2$

$xy - x = 3y + 2$

$xy - 3y = x + 2$

$y(x-3) = x + 2$

$y = \frac{x+2}{x-3}$

4) $\therefore f^{-1}(x) = \frac{x+2}{x-3}$

5) Since the inverse is a function (it does not have any $\pm$) we do not need to impose a restriction on the domain.

Strategy: Finding Inverses Algebraically

Example:

a) Find the inverse function of $f(x) = \sqrt[3]{x}$

Solution:

1. $y = \sqrt[3]{x}$
2. $x = \sqrt[3]{y} \Rightarrow \; y = x^3$
3. $f^{-1}(x) = x^3$

b) Find the inverse function of $f(x) = \frac{1}{2}x - \frac{7}{2}$

Solution:

1. $y = \frac{1}{2}x - \frac{7}{2}$
2. $x = \frac{1}{2}y - \frac{7}{2}$
3. $x + \frac{7}{2} = \frac{1}{2}y \Rightarrow y = 2x + 7$
4. $f^{-1}(x) = 2x + 7$

Check: $f(1) = \frac{1}{2}1 - \frac{7}{2} = -3; \quad f^{-1}(-3) = 2(-3) + 7 = 1$

Examples: Finding Inverses Algebraically

Determine the inverse function of the following: $f(x) = 2x^2 + 8x - 4$

Solution:

1) $y = 2x^2 + 8x - 4$

2) $x = 2y^2 + 8y - 4$

3) $x = 2(y^2 + 4y) - 4$

(Completing the square, $4 \div 2$ then square gives $+4$)

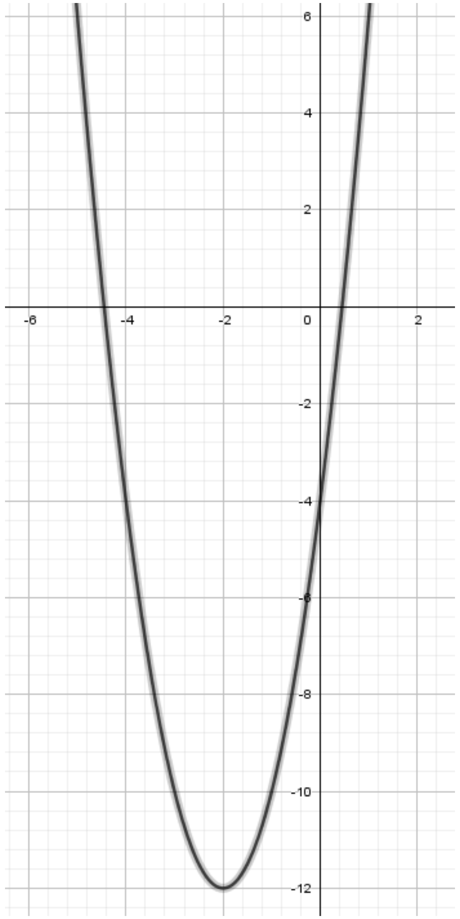
$$
\begin{aligned}
x &= 2(y^2 + 4y + 4 - 4) - 4 \\
x &= 2(y^2 + 4y + 4) - 12 \\
x &= 2(y + 2)^2 - 12 \\
x + 12 &= 2(y + 2)^2 \\
\frac{x+12}{2} &= (y + 2)^2
\end{aligned}
$$

$$\pm\sqrt{\frac{x+12}{2}} = y + 2$$

$$\pm\sqrt{\frac{x+12}{2}} - 2 = y$$

4) $\therefore y = \pm\sqrt{\frac{x+12}{2}} - 2$

5) Since this is not a function (as it has $\pm$), we must restrict the domain of the original function. We note that if we complete the square on the original, we get: $y = 2(x + 2)^2 - 12$. This is a parabola opening up with a vertex at $(-2,12)$:



$\therefore$ If we restrict the domain of $f(x)$ to $(-\infty, -2]$ we will get the inverse function to be $f^{-1}(x) = -\sqrt{\frac{x+12}{2}} - 2$

Or we can restrict the domain of $f(x)$ to $[-2, \infty)$ to get the inverse function to be $f^{-1}(x) = \sqrt{\frac{x+12}{2}} - 2$

Strategy: Finding Range using Inverse Functions

How To Use it:	When To Use it:	Why this works?
To find the range of a function $f(x)$, you can: 1) Find the inverse function $f^{-1}(x)$ 2) Identify the domain of $f^{-1}(x)$, as this will be the range of $f(x)$.	When you want to find the range of a function and the inverse function can be found.	Finding an inverse function means to switch the input (x) and the output (y). This means that it switches the domain and range. Since finding the domain of a function is a lot easier to spot, finding the inverse can help find the range.

Determine the domain and range of the following: $f(x) = \frac{3x+2}{x-1}$

Solution:

We note that the domain is easy to spot, as we simply cannot have any divisions by 0. In this case, we see that $x \neq 1$ which means that the domain is $D = (-\infty, 1) \cup (1, \infty)$.

To find the range, we can simply find the inverse function rather than graphing the function. We note this was our previous example, so the inverse function we found earlier was: $\therefore f^{-1}(x) = \frac{x+2}{x-3}$.

The domain of the inverse function is easy to calculate as we cannot divide by 0 which means that $x \neq 3$. This means the domain of the inverse is $D = (-\infty, 3) \cup (3, \infty)$ which is also equal to the range of the original function.